Exercises for 'Functional Analysis 2' [MATH-404]

(26/05/2025)

Ex 13.1 (Equivalent statements of Brouwer's fixed point theorem*)
Let $\overline{B}_1 := \overline{B_1(0)} \subset \mathbb{R}^n$ be the closed unit ball. Show that the following claims are equivalent :

1) Every continuous map $f : \overline{B}_1 \to \overline{B}_1$ has a fixed point.
2) There exists no continuous map $R : \overline{B}_1 \to \partial B_1$ such that $R(x) = x$ for all $x \in \partial B_1$.
3) Every continuous function $v : \overline{B}_1 \to \mathbb{R}^n$ such that $\langle v(x), x\rangle \leq 0$ for all $x \in \partial B_1$ has a zero in $\overline{B_1(0)}$.

Hint: The implication 2) $\implies$ 1) has been proven in the course. Prove 1) $\implies$ 3) and 3) $\implies$ 2).

Ex 13.2 [Not examinable] (An alternative extension construction)
Let $K \subset \mathbb{R}^n$ be a nonempty, compact and convex set and $f : K \to K$ be continuous. In the lecture we constructed a continuous extension $\widetilde{f} : \mathbb{R}^n \to K$. In this exercise we review a different construction, proposed by a student to Matthias Ruf during the break of the online lectures in 2021.

a) Show that for $x \in \mathbb{R}^n$ there exists a unique point $k(x) \in K$ such that $|x - k(x)| = \inf_{k\in K} |x - k|$.
   Hint: For the uniqueness, use that you can equivalently minimize the function $|x - k|^2$ with respect to $k \in K$ and that this function is strictly convex.
b) Show that the map $k : \mathbb{R}^n \to K$, $x \mapsto k(x)$ is continuous.
   Hint: Consider $x_j \to x$. Show that any converging subsequence of $k(x_j)$ converges to a minimizer of $k \mapsto |x - k|$, using the minimality of $k(x_j)$.
c) Show that the map $\widetilde{f}(x) = f(k(x))$ for $x \in \mathbb{R}^n$ defines a continuous extension of f to $\mathbb{R}^n$ such that $\widetilde{f}(\mathbb{R}^n) \subset K$. Can you replace compactness of K by a weaker assumption ?

Solution 13.2 : a) To prove existence it suffices to note that K is closed. Indeed, any minimizing sequence will be bounded and since its distance to x is bounded. By continuity of the function $k \mapsto |x - k|$, any limit of a minimizing sequence will be a minimizer since it belongs to K. To show uniqueness, assume that $k_1, k_2 \in K$ are different minimizers of $K \ni k \mapsto |x - k|$. Then they also minimize $K \ni k \mapsto |x - k|^2$. By strict convexity of this map we deduce that

$$|x - \tfrac{1}{2}k_1 + \tfrac{1}{2}k_2|^2 < \tfrac{1}{2}|x - k_1|^2 + \tfrac{1}{2}|x - k_2|^2,$$

which contradicts the minimality since by convexity of K we have that $\frac{1}{2}k_1 + \frac{1}{2}k_2 \in K$. Finally, observe that $k(x) = x$ for $x \in K$.

Remark: One can show that on Euclidean spaces, the only sets with a unique closed point projection are closed, convex sets. In Hilbert spaces this problem seems to be still open.

b) Let $x_j \in \mathbb{R}^n$ be such $x_j \to x$. Note that $|k(x_j)| \leq |k(x_j) - x_j| + |x_j| \leq |k(x) - x_j| + |x_j| \leq |k(x)| + 2|x_j|$. Hence $k(x_j)$ is a bounded sequence [1] and up to a subsequence we can assume

1. Assuming compactness of K, this is trivial. But we will show that the proof only requires closedness.

that $k(x_j) \to k_\infty \in K$, where we used that K is closed. We claim that $k_\infty = k(x)$. Since this result is independent of the subsequence, this proves that k is continuous in x. Note that for all $k_0 \in K$ we have

$$|x - k_\infty| = \lim_{j\to+\infty} |x_j - k(x_j)| = \lim_{j\to+\infty} \inf_{k\in K} |x_j - k| \le \lim_{j\to+\infty} |x_j - k_0| = |x - k_0|.$$

Hence by definition $k_\infty = k(x)$.

c) As noted in a), it holds that $k(x) = x$ for $x \in K$. Hence $\widetilde{f}$ is an extension of f. Moreover, by the continuity of $x \mapsto k(x)$, it follows that $\widetilde{f}$ is continuous and by construction we have $\widetilde{f}(\mathbb{R}^n) \subset K$. Note that we only used the fact that K is convex and closed. Compactness can therefore be weakened.

Ex 13.3 (Counterexample to Brouwer's fixed point theorem in infinite dimensions)
Let ℓ^2 be the Banach space of square-summable, real-valued sequences, i.e., $x = (x_i)_{i\in\mathbb{N}} \in \ell^2$ if and only if $x_i \in \mathbb{R}$ and $\|x\|_2^2 := \sum_{i\ge1} x_i^2 < +\infty$. Set $D = \{x \in \ell^2 : \|x\|_2 \le 1\}$ and define $f : D \to \ell^2$ by

$$f(x) = (\sqrt{1 - \|x\|_2^2}, x_1, x_2, x_3, \ldots).$$

Show that $f(D) \subset D$, f is continuous, but has no fixed point.

Solution 13.3 : First note that $f(x) \in \ell^2$ since the square-root is real-valued for $x \in D$ and moreover

$$\|f(x)\|^2 = 1 - \|x\|_2^2 + \sum_{i\ge1} x_i^2 = 1 - \|x\|_2^2 + \|x\|_2^2 = 1.$$

In particular, $f(D) \subset D$. Moreover, for $x, y \in D$ it holds that

$$\|f(x) - f(y)\|_2^2 = \sqrt{1 - \|x\|_2^2} - \sqrt{1 - \|y\|^2} + \sum_{i\ge1} |x_i - y_i|^2 = \sqrt{1 - \|x\|_2^2} - \sqrt{1 - \|y\|^2} + \|x - y\|_2^2.$$

Using that the square-root and the norm are continuous, it follows that f is continuous on D. Finally, we prove that f has no fixed point. If $f(x) = x$, then by iteration

$$x_n = f(x)_n = x_{n-1} = \ldots = f(x)_1 = \sqrt{1 - \|x\|_2^2}.$$

Hence, as a sequence, x is constant. Since we assume it is square-summable, we obtain that $x = 0$. But then $\sqrt{1 - \|x\|_2^2} = 0$, which yields a contradiction.

Ex 13.4 (Properties of the subdifferential)
Let $E\colon H \to \mathbb{R}_+ \cup \{\infty\}$ be a convex and lower semicontinuous functional on a real Hilbert space H. We view its (possibly empty, possibly multivalued) subdifferential as a map $\partial^- E\colon H \to 2^H$. Here 2^H denotes the power set of H.

a) Show $\partial^- E$ is a monotone operator. That is, for every $x, y \in H$, every $x^* \in \partial^- E(x)$, and every $y^* \in \partial^- E(y)$,
$$\langle y^* - x^*, y - x\rangle \ge 0.$$

b) Show the following strong-weak closedness property of the graph of $\partial^- E$. Assume $(x_n)_{n\in\mathbb{N}}$ is a sequence in H which converges to $x \in H$. Moreover, let $(x_n^*)_{n\in\mathbb{N}}$ be a sequence of elements $x_n^* \in \partial^- E(x_n)$ weakly converging to $x^* \in H$. Then $x^* \in \partial^- E(x)$.

Solution 13.4 : a) Note $x, y \in \mathscr{D}(E)$ by assumption. Using the defining properties of the inclusions $x^* \in \partial^- E(x)$ and $y^* \in \partial^- E(y)$,

$$\begin{aligned}\langle x^*, y - x\rangle &\leq E(y) - E(x),\\ \langle y^*, x - y\rangle &\leq E(x) - E(y).\end{aligned}$$

Summing these two inequalities yields

$$\langle x^* - y^*, y - x\rangle = \langle x^*, y - x\rangle + \langle y^*, x - y\rangle \leq 0.$$

b) Recall that on every Hilbert space, the scalar product of a strongly convergent against a weakly convergent sequence converges to the scalar product of the respective limits. Combining this statement with the lower semicontinuity of E and the inclusion $x_n^* \in \partial^- E(x_n)$ yields for every $y \in H$ that

$$E(x) + \langle x^*, y - x\rangle \leq \liminf_{n\to\infty} E(x_n) + \lim_{n\to\infty} \langle x_n^*, y - x_n\rangle \leq E(y).$$

By taking $y \in \mathscr{D}(E)$, combined with nonnegativity of E this shows $x \in \mathscr{D}(E)$. In turn, we obtain $x^* \in \partial^- E(x)$.